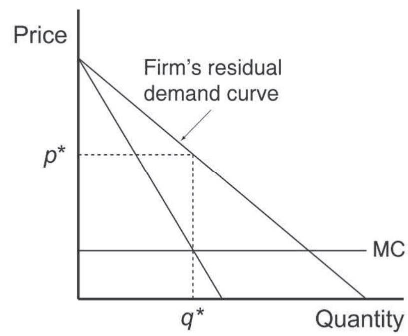
A LITTLE ALGEBRAIC INSIGHT ON PROFIT MAXIMIZATION

Let p^* and q^* be the profit-maximizing price and the associated profit-maximizing quantity. (Remember that prices and quantities are related to one another through the firm's (residual) demand curve as shown in the diagram.)



If fixed costs are zero and marginal cost is a constant c, then profits π^* are:

$$\pi^* = p^* q^* - cq^* \tag{1}$$

Let Δp be a positive increase in price and Δq be the associated reduction in the profit-maximizing quantity, so that $p + \Delta p$ is the new price and $q - \Delta q$ is the new quantity.

Since by definition p^* and q^* are the profit-maximizing price and quantity, the firm's profits at these prices must be at least equal to if not greater than the firm's profits at any other price, that is,

$$p^* q^* - cq^* \geq (p^* + \Delta p)(q^* - \Delta q) - c(q^* - \Delta q). \tag{2}$$

Expanding the right-hand side of Equation 2 yields:

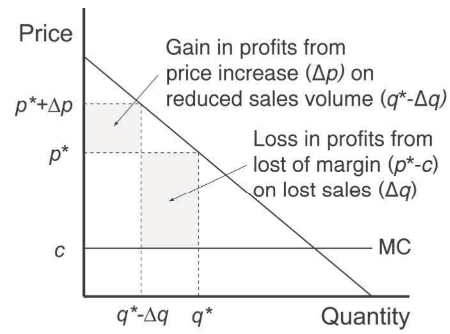
$$p^* q^* - cq^* \geq p^* q^* - p^* \Delta q + q^* \Delta p - \Delta p \Delta q - cq^* + c\Delta q. \tag{3}$$

Cancelling terms that appear on both sides of the equation

$$0 \geq -p^* \Delta q + q^* \Delta p - \Delta p \Delta q + c\Delta q \tag{4}$$

and rearranging

$$0 \geq +\Delta p(q^* - \Delta q) - \Delta q(p^* - c). \tag{5}$$



Equation 5 has an important and straightforward interpretation: since the right-hand side is no more than zero, the gain in profits resulting from the price increase on the decreased quantity of sales $\left(\Delta p\left(q^{*}-\Delta q\right)\right)$ can be no more than the loss of margin on the lost sales $\left(\Delta q\left(p^{*}-c\right)\right)$. At best, the firm breaks even on the price increase and it could well lose money. The relationship between the gain in profits resulting from a price increase and the loss in profit resulting from the loss of margin from lost sales constantly arises in applied antitrust analysis.

A few more steps and we will show that at the profit-maximizing price and quantity marginal cost equals marginal revenue. Expanding the last term on the right-hand side of Equation 5 gives

$$0 \geq \Delta p\left(q^{*}-\Delta q\right)-p^{*} \Delta q+c \Delta q \tag{6}$$

or equivalently

$$-\Delta p\left(q^{*}-\Delta q\right)+p^{*} \Delta q \geq c \Delta q. \tag{7}$$

The left-hand side of Equation 7 is the incremental net revenue the firm would receive if it expanded its output by Δq and dropped its price to Δp (you can see this from the diagram at the top of the page). When Δq is very small, the expression on the left-hand side is marginal revenue. The right-hand side of Equation 7 is the additional (marginal) cost of producing Δq. Equation 7 says that when prices are above the profit-maximizing price—and output below the profit-maximizing output—the firm's marginal revenue is equal to or greater than the firm's marginal cost. We can capture this in Equation 8:

$$MR_{p^{+}} \geq MC_{p^{+}} \tag{8}$$

where MR is marginal revenue and MC is marginal cost, and the subscript p^{+} indicates that the firm is pricing above its profit-maximizing price. If the firm's marginal revenue is strictly greater than the firm's marginal costs, then the firm increases its profits when it drops its price and expands its output.

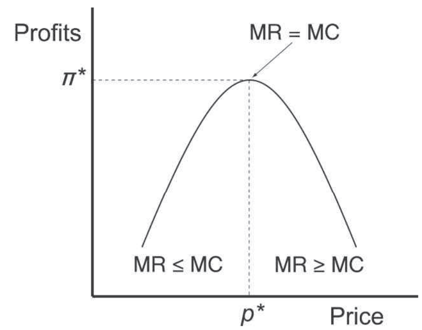
We derived Equation 7 assuming that the firm increased its price and reduced its output compared to the profit-maximizing price and output. If we make the opposite assumption, then the counterpart of Equations 7 and 8 are

$$-\Delta p\left(q^{*}-\Delta q\right)+p^{*} \Delta q \leq c \Delta q. \tag{9}$$

and

$$MR_{p^-} \leq MC_{p^-}. \tag{10}$$

Equation 10 says that when prices are below the profit-maximizing price, marginal revenue is no greater than marginal cost. If marginal revenue is strictly less than marginal cost, the firm is losing money on the marginal sales and will make more profits if it decreases its output and raises its price.



What is the relationship between marginal revenue and marginal cost at the firm's profit-maximizing price and output? A profit-maximizing price and output must satisfy both Equations 8 and 10, so that a change in price (or output) in either direction is profit losing. So at the profit-maximizing price and output

$$MR = MC. \tag{11}$$

Q.E.D. (loosely speaking)