# A Profit-Maximization Problem Step-by-Step (by Step)

A firm faces a residual demand curve of $q = 100 - \frac{1}{2}p$ and has constant marginal costs of $mc = 4$ and no fixed costs.

1. The firm is considering choosing a production level $q = 20$. Is this the profit-maximizing quantity for the firm?
2. What is the profit-maximizing quantity and price of the firm? What profits will it earn?
3. How do the profit-maximizing quantity, price, and profits of the firm compare to the competitive (aggregate) quantity, price, and (aggregate) profits?

## Answers (using lots of words)

### ANSWER Q1

**1. The firm is considering choosing a production level *q* = 20. Is this the profit-maximizing quantity for the firm?**

*Strategy*: Find the firm's marginal profits at $q$ =20. If marginal profits are positive, then the firm would be producing too little. If marginal profits are negative, then the firm would be producing too much.

Marginal profits ($m\pi$) are equal to marginal revenue ($mr$) minus marginal costs ($mc$):

$$m\pi = mr - mc. \tag{1}$$

*First*, find the formula for the marginal revenue curve in order to determine marginal revenue at $q = 20$. To do this, convert the demand curve into an inverse demand curve (so that $p$ is on the left-hand side):

$$q = 100 - \frac{1}{2}p. \tag{2}$$

Rearrange Equation 2 so that $p$ is on the left-hand side:

$$\frac{1}{2}p = 100 - q. \tag{3}$$

Multiply both sides of Equation 3 by 2 to create the *inverse demand curve*:

$$p = 200 - 2q. \tag{4}$$

We know that when the inverse demand curve has the form $p = a + bq$, the marginal revenue curve has the form $mr = a + 2bq$. Given Equation 4, we see that $a = 200$ and $b = -2$. Therefore, the *marginal revenue curve* of the firm is:

$$\begin{aligned} mr &= 200 + 2 \times (-2) \times q \\ &= 200 - 4q \end{aligned} \tag{5}$$

When $q$ = 20.

$$\begin{aligned} mr &= 200 - 4 \times 20 \\ &= 200 - 80 \\ &= 120. \end{aligned} \tag{6}$$

*Second*, find the firm's marginal cost. We know from the hypothetical that $mc$ = 4.

*Third*, calculate marginal profit. We know that $mr$ = 120 and $mc$ = 4, so:

$$\begin{aligned} m\pi &= mr - mc \\ &= 120 - 4 \\ &= 116. \end{aligned} \tag{7}$$

Since marginal revenue at $q$ = 20 is positive, this means that the firm could earn additional profits by increasing its production level. So $q$ = 20 is not the profit-maximizing quantity.

**ANSWER Q2**

**2. What is the profit-maximizing quantity and price of the firm? What profits will it earn?**

*Strategy*: A firm maximizes its profit by setting its production level $q^*$ so that its marginal revenue equals its marginal costs. First, find the marginal revenue formula, set it equal to marginal costs, and solve for $q^*$. Second, knowing $q^*$, use the demand curve to find the market-clearing price $p^*$. Third, since profits are revenue minus costs, calculate the firm's profit-maximizing level of revenues $r^*$ by calculating revenues at $q^*$ and subtracting the costs to produce $q^*$.

*First*, find the firm's profit-maximizing production level $q^*$. We know that a profit-maximizing firm sets its production level $q^*$ so that its marginal profits are zero (at the top of the hill of the profit curve). This means that its sets its production level so that its marginal revenues equals its marginal costs.[1]

$$mr = mc. \tag{8}$$

Substituting the formula for marginal revenue from Equation 5 and $mc$ = 4 into Equation 8 gives:

$$200 - 4q^* = 4, \tag{9}$$

Where $q^*$ is the firm's profit-maximizing quantity. Solving for $q^*$ gives:

$$\begin{aligned} 4q^* &= 200 - 4 \\ &= 196 \end{aligned} \tag{10}$$

Solving Equation 10 for q* gives:

$$q^* = \frac{196}{4} = 49.$$

So the firm's profit-maximizing level of production is $q^*$ = 49.

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[1] If $m\pi = mr - mc = 0$, then $mr = mc$. This is known as the *first-order condition* for a profit maximum.

*Second*, find the firm's profit-maximizing price $p^*$ by substituting $q^* = 49$ into the inverse demand function (Equation 4):

$$\begin{aligned} p^* &= 200 - 2q^* \\ &= 200 - 2 \times 49 \\ &= 200 - 98 \\ &= 102. \end{aligned} \tag{11}$$

So the market-clearing price when the firm produces its profit-maximizing level of output will be 102.

Third, find the firm's profits $\pi^*$ at its profit-maximizing production level $q^*$. Profits $\pi$ are equal to revenues $r$ minus costs $c$:

$$\pi = r - c. \tag{12}$$

Since revenues are equal to price times quantity, the firm's revenues at $q^* = 49$ are:

$$\begin{aligned} r &= pq \\ &= 102 \times 49 \\ &= 4998. \end{aligned} \tag{13}$$

Costs are equal to fixed costs ($f$) plus variable cost ($v$):

$$c = f + v. \tag{14}$$

The firm has fixed costs $f = 0$. Since it has constant marginal costs, variable costs are equal to marginal cost times quantity:

$$v = mc \times q. \tag{15}$$

So when $q = 49$, the firm's costs $c$ are:

$$\begin{aligned} c &= f + mc \times q \\ &= 0 + 4 \times 49 \\ &= 196. \end{aligned} \tag{16}$$

Subtracting the firm's revenues from the firm's costs gives the firm's profits:

$$\begin{aligned} \pi^* &= r^* - c^* \\ &= 4998 - 196 \\ &= 4802. \end{aligned} \tag{17}$$

So the answer to Question 2 is:

$$\begin{aligned} q^* &= 49 \\ p^* &= 102 \\ \pi^* &= 4802. \end{aligned} \tag{18}$$

## ANSWER Q3

**3. How do the profit-maximizing quantity, price, and profits of the firm compare to the competitive quantity, price, and (aggregate) profits?**

*Strategy*: In a competitive market, firms set their production level so that price is equal to marginal cost. Calculate the aggregate competitive quantity $Q_c$ by setting $Q_c$ equal to marginal cost. Calculate the market-clearing price $p_c$ at aggregate production level $Q_c$ from the inverse demand curve. Calculate the aggregate competitive profits by calculating the total revenues earned and subtracting the total costs incurred to produce production level $Q_c$.

*First*, calculate the competitive quantity $Q_c$ by setting $Q_c$ equal to marginal cost:

$$p_c = mc \tag{19}$$

From the hypothetical, $mc = 4$, so $p_c = 4$.

*Second*, calculate the aggregate quantity demand at $p_c = 4$ from the demand curve (Equation 2):

$$\begin{aligned} Q_c &= 100 - \frac{1}{2} p_c \\ &= 100 - \frac{1}{2} \times 4 \\ &= 100 - 2 \\ &= 98. \end{aligned} \tag{20}$$

*Third*, calculate the aggregate profits by determining total revenues and subtracting total costs.

$$\begin{aligned} \pi_c &= r_c - c_c \\ &= p_c Q_c - mc \times Q_c \\ &= 4 \times 98 - 4 \times 98 \\ &= 0. \end{aligned} \tag{21}$$

So in a competitive market, the firms earn aggregate profits of zero (which makes sense since the price they receive for each unit of production is equal to the cost of producing that unit).

In sum,

$$\begin{aligned} p_c &= 4 \\ Q_c &= 98 \\ \pi_c &= 0. \end{aligned} \tag{22}$$

Comparing with the monopoly case:

| | Monopoly market | Competitive market |
|---|---|---|
| Price | 102 | 4 |
| Aggregate quantity | 49 | 98 |
| Total profits | 4802 | 0 |

**Answers (using very few words)**

## ANSWER Q1

**1. The firm is considering choosing a production level *q* = 20. Is this the profit-maximizing quantity for the firm?**

$$q = 100 - \frac{1}{2}p \Rightarrow p = 200 - 2q$$
$$r = pq = [200 - 2q]q = 200q - 2q^2$$
$$mr = \frac{dr}{dq} = 200 - 4q$$

At $q$ =20, $mr$ = 120. Since $mr$ > 0, the firm is producing too little.

## ANSWER Q2

**2. What is the profit-maximizing quantity and price of the firm? What profits will it earn?**

The first order condition for a profit-maximizing firm is that $mr = mc$. Using the marginal revenue formula from Answer 1 and $mc$ = 4 from the hypothetical:

$$mr = \frac{dr}{dq} = 200 - 4q^* = 4 = mc$$

So $q^*$ = 49.

From the inverse demand function in Answer 1:

$$\begin{aligned} p^* &= 200 - 2q^* \\ &= 200 - (2)(49) \\ &= 102. \end{aligned}$$

So:

$$\begin{aligned} \pi^* &= r^* - c^* \\ &= p^* q^* - (mc) q^* \\ &= (102)(49) - (4)(49) \\ &= 4802. \end{aligned}$$

## ANSWER Q3

**3. How do the profit-maximizing quantity, price, and profits of the firm compare to the competitive quantity, price, and (aggregate) profits?**

In competitive markets, $p_c = mc$, so $p_c$ =4. From the demand curve, $Q_c = 100 - 0.5p_c$ =98. Total profits:

$$\pi_c = p_c Q_c - (mc) Q_c = [p_c - mc] Q_c = 0 \text{ since } p_c = mc.$$